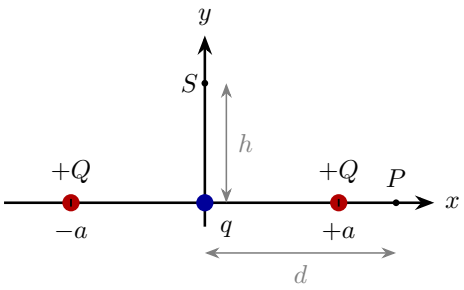
1. Two identical charges $+Q$ are fixed on the x-axis at positions $x = -a$ and $x = +a$. A third charge q is placed at the origin, as shown in the figure. Let $V = 0$ at infinity throughout this problem.



[6 Pts.] (a) What should the charge q be so that the net force on *every* charge in the system is zero?

By symmetry, the net force on q at the origin is always zero (the two $+Q$ push symmetrically). The constraint comes from requiring the net force on each $+Q$ to vanish.

- Repulsion from $+Q$ at $x = -a$ (distance $2a$), directed $+x$: $F_1 = \dfrac{kQ^2}{4a^2}$
- Attraction toward q at origin (distance a), directed $-x$ if $q < 0$: $F_2 = \dfrac{k|q|Q}{a^2}$

Setting $F_1 = F_2$:

$$\frac{kQ^2}{4a^2} = \frac{k|q|Q}{a^2} \implies |q| = \frac{Q}{4}$$

Since q must attract $+Q$ inward, q is negative: $q = -\frac{Q}{4}$

[6 Pts.] (b) Using the value of q found in part (a), find the electric potential V at point P on the positive x-axis at a distance d from the origin ($d > a$). Express your answer in terms of Q, a, and d.

Distances from each charge to P: $r_{+Q,\text{left}} = d + a$, $\quad r_{+Q,\text{right}} = d - a$, $\quad r_q = d$.

Potential is a scalar sum:

$$V = kQ\left(\frac{1}{d+a} + \frac{1}{d-a} - \frac{1}{4d}\right)$$

[7 Pts.] (c) Find the electric potential V at point S on the y-axis at a height h above the origin (i.e. at position $(0, h)$). Express your answer in terms of Q, a, and h.

Distance from each $+Q$ to S: $r = \sqrt{a^2 + h^2}$. Distance from q to S: h.

$$V = kQ\left(\frac{2}{\sqrt{a^2+h^2}} - \frac{1}{4h}\right)$$

[6 Pts.] (d) Find the total electric potential energy U of the three-charge system. Express your answer in terms of Q and a.

Pair	Distance	Energy
$(+Q,\ +Q)$	$2a$	$\dfrac{kQ^2}{2a}$
$(+Q_{\text{left}},\ q)$	a	$-\dfrac{kQ^2}{4a}$
$(+Q_{\text{right}},\ q)$	a	$-\dfrac{kQ^2}{4a}$

$$\Rightarrow U = \frac{kQ^2}{2a} - \frac{kQ^2}{4a} - \frac{kQ^2}{4a} = 0$$

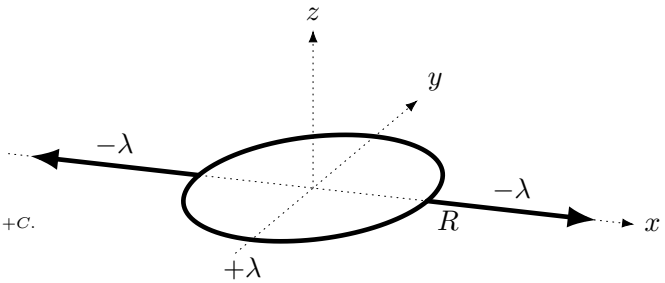
2. A thin circular ring of radius R, centered at the origin and lying in the x–y plane, carries uniform linear charge density $+\lambda$, where $\lambda > 0$.

   Two semi-infinite straight wires lie along the x-axis. One extends from $x = R$ to $+\infty$, and the other from $x = -R$ to $-\infty$. Both carry uniform linear charge density $-\lambda$.

The following indefinite integrals may be useful ($b > 0$):

$$\int \frac{dx}{\sqrt{x^2+b^2}} = \ln\left(\frac{x+\sqrt{x^2+b^2}}{b}\right) + C,$$

$$\int \frac{dx}{x^2+b^2} = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+C, \quad \int \frac{dx}{(x^2+b^2)^{3/2}} = \frac{x}{b^2\sqrt{x^2+b^2}}+C.$$



[5 Pts.] (a) Use symmetry to determine which components of the electric field vanish at a point $(0, 0, z)$.

Ring: For every element dq at $(x, y, 0)$, the opposite element at $(-x, -y, 0)$ cancels E_x and E_y; z-components add.
Wires: Elements at $\pm x_0$ cancel E_x (opposite horizontal displacements, same charge density) and $E_y = 0$ (wires lie on x-axis). z-components add.

$$E_x = E_y = 0; \quad \text{only } E_z \text{ survives.}$$

[12 Pts.] (b) Calculate the total electric field $\mathbf{E}(0, 0, z)$.

Ring contribution. Each element dl is at distance $\sqrt{R^2+z^2}$ from $(0, 0, z)$. The axial component:

$$dE_z = \frac{\lambda\, dl}{4\pi\epsilon_0} \cdot \frac{z}{(R^2+z^2)^{3/2}}$$

Integrating $\oint dl = 2\pi R$: $\Rightarrow E_z^{\text{ring}} = \frac{\lambda R z}{2\epsilon_0(R^2+z^2)^{3/2}}$

Wire contribution.

For Wire 1 (x from R to ∞, charge $-\lambda$): $dE_z = \frac{-\lambda\, dx}{4\pi\epsilon_0} \cdot \frac{z}{(x^2+z^2)^{3/2}}$

Using $\displaystyle\int_R^\infty \frac{dx}{(x^2+z^2)^{3/2}} = \frac{1}{z^2}\left(1 - \frac{R}{\sqrt{R^2+z^2}}\right)$:

$$E_z^{\text{wire 1}} = \frac{-\lambda}{4\pi\epsilon_0 z}\left(1 - \frac{R}{\sqrt{R^2+z^2}}\right)$$

Wire 2 gives the same contribution by symmetry, so: $E_z^{\text{wires}} = \frac{-\lambda}{2\pi\epsilon_0 z}\left(1 - \frac{R}{\sqrt{R^2+z^2}}\right)$

Total:

$$E_z = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{\pi R z}{(R^2+z^2)^{3/2}} - \frac{1}{z}\left(1 - \frac{R}{\sqrt{R^2+z^2}}\right)\right], \quad \mathbf{E} = E_z\hat{z}$$

[8 Pts.] (c) Find all points on the z-axis where the electric field is zero. Set $E_z = 0$:

$$\frac{\pi R z}{(R^2+z^2)^{3/2}} = \frac{1}{z}\left(1 - \frac{R}{\sqrt{R^2+z^2}}\right)$$

Let $u = \dfrac{R}{\sqrt{R^2+z^2}}$, $0 < u < 1$, $z = \dfrac{R\sqrt{1-u^2}}{u}$.

Then:

$$\frac{Rz}{(R^2+z^2)^{3/2}} = \frac{u^2\sqrt{1-u^2}}{R}, \qquad \frac{1}{z}(1-u) = \frac{u(1-u)}{R\sqrt{1-u^2}}$$

Substituting and simplifying (dividing both sides by $u\sqrt{1-u}$):

$$\pi u\sqrt{1+u} = \frac{1}{\sqrt{1+u}} \implies \pi u(1+u) = 1 \implies \pi u^2 + \pi u - 1 = 0$$

Positive root:

$$u = \frac{-\pi + \sqrt{\pi^2+4\pi}}{2\pi} = \frac{-1+\sqrt{1+4/\pi}}{2} \approx 0.2539$$

Converting back to z:

$$z = \frac{R\sqrt{1-u^2}}{u} \approx \frac{R \times 0.9673}{0.2539}$$

By symmetry (E_z is odd in z), both $\pm z_0$ are solutions:

$$z \approx \pm 3.81\, R$$

There are exactly two zero-field points, symmetric about the xy-plane.

3. A nonconducting sphere of radius r_0 has a spherical cavity of radius r_1 centered at the sphere's center. A total charge Q is distributed uniformly in the "shell" (between $r > r_1$ and $r < r_0$).

[5 Pts.] (a) What is the expression for the volume charge density ρ for $r_1 < r < r_0$?

Shell volume:

$$V_{\text{shell}} = \tfrac{4}{3}\pi(r_0^3 - r_1^3) \qquad \Rightarrow \rho = \frac{3Q}{4\pi(r_0^3 - r_1^3)}$$

[12 Pts.] (b) Find the expression for the magnitude of the electric field as a function of r for $r < r_1$,$r_1 < r < r_0$, and $r > r_0$.

Apply Gauss's law: $E \cdot 4\pi r^2 = Q_{\text{enc}}/\epsilon_0$.

Region 1: $r < r_1$ **(cavity)**

$Q_{\text{enc}} = 0$: $\quad E = 0$

Region 2: $r_1 < r < r_0$ **(inside shell)**

$$Q_{\text{enc}}(r) = \rho \cdot \tfrac{4}{3}\pi(r^3 - r_1^3) = \frac{Q(r^3 - r_1^3)}{r_0^3 - r_1^3}$$

$$E = \frac{Q(r^3 - r_1^3)}{4\pi\epsilon_0\, r^2(r_0^3 - r_1^3)}, \quad \text{radially outward}$$

Check: At $r = r_1$: $E = 0$ ✓; at $r = r_0$: $E = \dfrac{Q}{4\pi\epsilon_0 r_0^2}$ ✓

Region 3: $r > r_0$ **(outside sphere)**

$Q_{\text{enc}} = Q$:

$$E = \frac{Q}{4\pi\epsilon_0 r^2}, \quad \text{radially outward}$$

[8 Pts.] (c) Find the potential difference $V(r_1) - V(r_0)$ between the inner and the outer surfaces of the shell for $r_0 = 3r_1$.

With $r_0 = 3r_1$: $r_0^3 - r_1^3 = 27r_1^3 - r_1^3 = 26r_1^3$.

$$V(r_1) - V(r_0) = \int_{r_1}^{r_0} E\, dr = \frac{Q}{4\pi\epsilon_0 \cdot 26r_1^3}\int_{r_1}^{3r_1}\left(r - \frac{r_1^3}{r^2}\right) dr$$

$$= \frac{Q}{4\pi\epsilon_0 \cdot 26r_1^3}\left[\frac{r^2}{2} + \frac{r_1^3}{r}\right]_{r_1}^{3r_1}$$

Evaluate limits:

$$\left(\frac{9r_1^2}{2} + \frac{r_1^2}{3}\right) - \left(\frac{r_1^2}{2} + r_1^2\right) = r_1^2\left(\frac{27 + 2 - 3 - 6}{6}\right) = \frac{20r_1^2}{6} = \frac{10r_1^2}{3}$$

$$V(r_1) - V(r_0) = \frac{Q}{4\pi\epsilon_0 \cdot 26r_1^3} \cdot \frac{10r_1^2}{3} = \frac{10Q}{312\pi\epsilon_0 r_1} = \frac{5Q}{156\pi\epsilon_0 r_1}$$

[12 Pts.] 4. (a) A spherical capacitor consists of two thin concentric spherical conducting shells, of radius r_1 and r_2, $r_2 > r_1$. The inner shell carries a uniformly distributed charge Q on its surface, and the outer shell an equal but opposite charge $-Q$. Determine the capacitance of the two shells. What would be the capacitance if $r_2 \to \infty$?

Inner shell radius r_1, outer shell radius r_2 ($r_1 < r_2$). Between the shells ($r_1 < r < r_2$):

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

Potential difference:

$$V_1 - V_2 = \int_{r_1}^{r_2} \frac{Q}{4\pi\epsilon_0 r^2}\, dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$C = \frac{Q}{V_1 - V_2} = 4\pi\epsilon_0 \frac{r_1 r_2}{r_1 - r_2}$$

As $r_2 \to \infty$: $C \to 4\pi\epsilon_0 r_1$ (isolated sphere) ✓

[10 Pts.] (b) Two spheres have radii a and b, and their centers are a distance d apart. Find the capacitance of this system provided $d \gg a, b$. *Hint: Because the spheres are far apart, assume the potential of each sphere equals the sum of potentials due to both charges, treating each as a point charge at its center.* Using the hint: Sphere 1 (charge $+Q$, radius a):

$$V_a = \frac{Q}{4\pi\epsilon_0 a} + \frac{(-Q)}{4\pi\epsilon_0 d} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{d}\right)$$

Sphere 2 (charge $-Q$, radius b):

$$V_b = \frac{Q}{4\pi\epsilon_0 d} + \frac{(-Q)}{4\pi\epsilon_0 b} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{d} - \frac{1}{b}\right)$$

Potential difference:

$$V_a - V_b = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{d}\right)$$

$$C = \frac{4\pi\epsilon_0}{\frac{1}{a} + \frac{1}{b} - \frac{2}{d}}$$

[3 Pts.] (c) As $d \to \infty$, can the two-sphere system be regarded as two capacitors connected in series or in parallel? Justify your answer. As $d \to \infty$, $2/d \to 0$:

$$C_\infty = \frac{4\pi\epsilon_0}{\frac{1}{a} + \frac{1}{b}}$$

Individual sphere capacitances: $C_a = 4\pi\epsilon_0 a$, $C_b = 4\pi\epsilon_0 b$.
Series combination: $\frac{1}{C_{\text{series}}} = \frac{1}{C_a} + \frac{1}{C_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{a} + \frac{1}{b}\right) \implies C_{\text{series}} = \frac{4\pi\epsilon_0}{\frac{1}{a}+\frac{1}{b}} = C_\infty$ ✓

The system behaves as two capacitors connected in **series**.